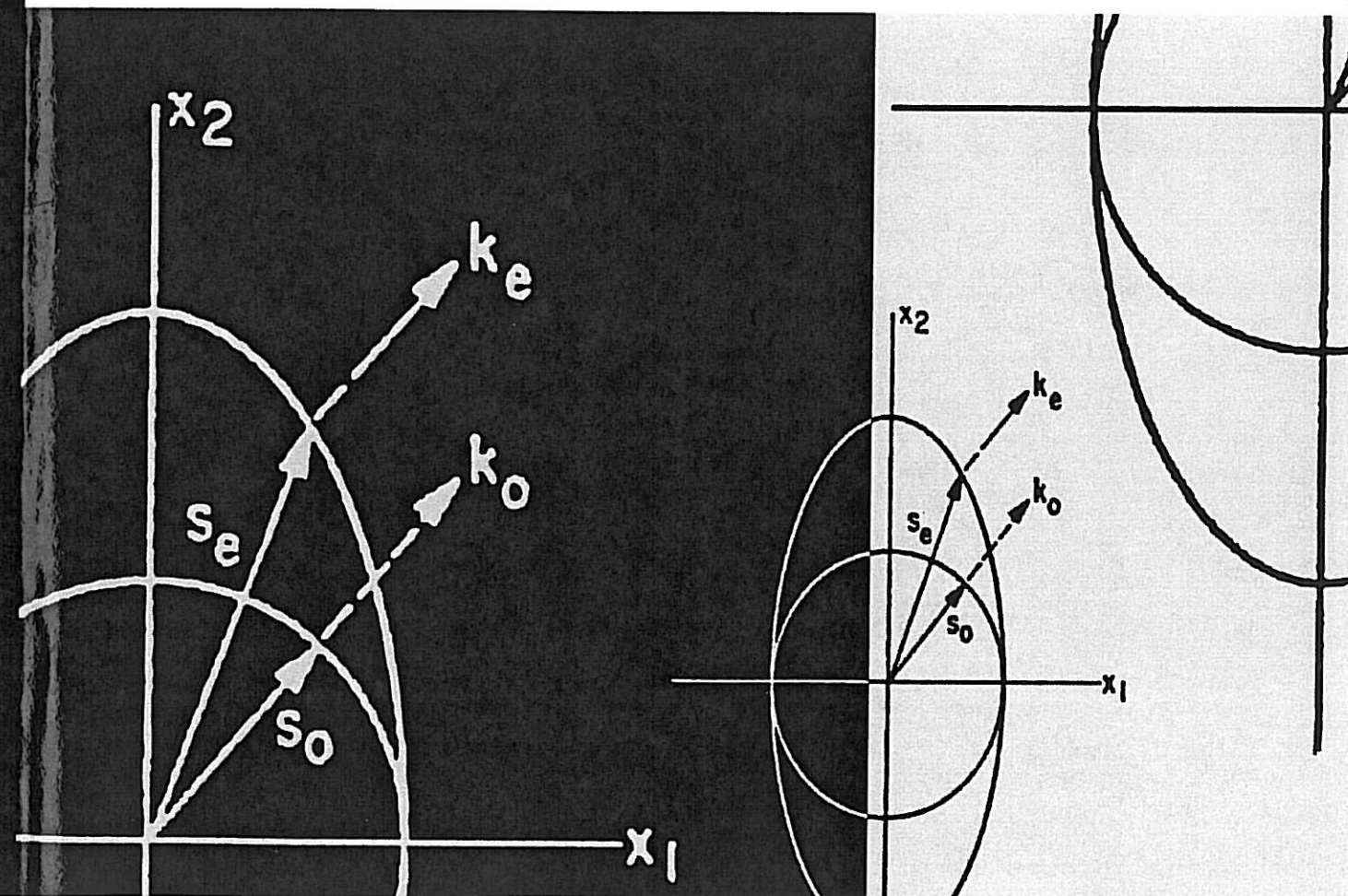


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Linear and Nonlinear Waves

G. B. Whitham



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**LINEAR AND
NONLINEAR WAVES**

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